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Q–Φ criticality in the extended phase space of $(n+1)$-dimensional RN-AdS black holes

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Abstract In order to achieve a deeper understanding of gravity theories, i.e., the quantum properties of gravity theories and the statistical explanation of gravitational entropy, it is important to further investigate the thermodynamic properties of a black hole at the critical point, besides the phase transition and critical behaviors. In this paper, by using Maxwell's equal area law, we choose T, Q, Φ as the state parameters and study the phase equilibrium problem of a general $(n+1)$-dimensional RN-AdS black holes thermodynamic system. The boundary of the two-phase coexistence region and its isotherm and isopotential lines are presented, which may provide a theoretical foundation for studying the phase transition and phase structure of black hole systems.

1 Introduction

In a previous analysis, the cosmological constant in AdS space-time and the state parameter-pressure in general thermodynamic system are always parallelized as

$$\Lambda = -\frac{n(n-1)}{2l^2}, \quad P = \frac{n(n-1)}{16\pi l^2} \tag{1}$$

and the corresponding thermodynamics volume for black hole thermodynamic system expresses as

$$V = \left(\frac{\partial M}{\partial P}\right)_{S,Q_i,J_k} \tag{2}$$

where T, P, and V are the variable state parameters [1–26]. This traditional method has been extensively applied to the characterization of general thermodynamic systems and also to the establishing of complete simulation for AdS space-time black hole thermodynamic liquid/gas system [27–45].

A more recent method is to choose state parameters (T, P, V) as variables and apply the Ehrenfest scheme to the study of various black hole critical phenomenon in AdS space-time. It has been proved that at the phase transition point, the second-order phase transition exists in AdS space-time black holes and the thermodynamics agree very well with Ehrenfest equation [12,15,17,35,36,46,47]. At present there are two common methods to investigate the critical phenomenon of AdS space-time Black Holes in the literature. The first is study the thermodynamics and state space geometry of black holes [18,19,25,26,37–39,44,48–50], which found that the black hole phase transition point meets the requirements of thermodynamic second-order phase transition. The second approach is to turn to Maxwell's equal area law and discuss the critical behavior of AdS space-time black hole system [19,51–55], which have also proved the existence of second-order phase transition at black hole phase transformation point.

Despite the promising results obtained about thermodynamic properties of AdS space-time black holes, from theoretical point of view, there should be critical behaviors and phase transition process, if taking black holes in AdS space-time as thermodynamic systems. However, the statistical mechanics background of Black Holes as thermodynamic systems is still unknown, which makes it very meaningful to study the relations between different thermodynamic properties of AdS space-time black holes. Moreover, such a study will contribute to a deeper understanding of the thermodynamic properties of black hole (entropy, temperature, and heat capacity), as well as the completion of self-consistent geometry theory of black hole thermodynamics.

The P–V phase diagram of AdS space-time black holes was analyzed in Ref. [30], which implied the existence of a mechanical unstable region when the black hole temperature is low. In this region with partial negative pressure, pressure increases together with volume $\partial M/\partial P > 0$ in the isotherm, which is similar to the result obtained from the P–V phase diagram of a Van der Waals–Maxwell liquid/gas. We have a possible solution to the famous Maxwell equal area

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law, which was extensively applied to the study of the AdS space-time black hole thermodynamic system [49,51–55]. The T–P curve yielded for the system;s biphase equilibrium and the slope expression of the biphase equilibrium curve showed that the AdS space-time black hole has the second-order phase transition and can be in a two-phase coexistence state. We remark here that, except for the phase transition point, other phase transitions in AdS space-time black hole are all of first order. However, all the above results were derived on the base of the precondition of invariance of electric charge. As is well known, the parameters describing the charged AdS space-time black hole thermodynamic system are not only related to the state parameters (T, P, V), but also electromagnetic parameters like the charge and electric potential. In this paper, we will use Maxwell's equal area law to study the thermodynamic properties of general $(n+1)$-dimensional RN-AdS black holes. More specifically, the discussion of the following two problems is the main motivation of our analysis: For a black hole thermodynamic system with constant P and V, is there still second-order phase transition when taking (T, Q, Φ) as state parameters? If so, is the critical point still the same as that when (T, P, V) are taken as state parameters? In the second part, we give a brief introduction to the $(n+1)$-dimensional RN-AdS black hole. In the third part, we apply Maxwell's equal area law to the case of an $(n+1)$-dimensional RN-AdS black hole thermodynamic system, and we obtain the relationship between different parameters and the boundary of a two-phase coexistence region. In this part, we mainly consider the canonical ensemble (with fixed charge) in the extended phase space. Moreover, we also add some discussion of the Gibbs free energy in the non-extended phase space in Sect. 4. The last part is for our conclusion.

2 General $(n+1)$-dimensional RN-AdS black holes

For general $(n+1)$-dimensional RN-AdS black holes, the space-time metric can be written as [39]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{n-1}^2, \tag{3}$$

where $d\Omega_{n-1}$ is the metric of the associated $(n+1)$-dimensional base manifold and

$$f(r) = k - \frac{8\Gamma(\frac{n}{2})M}{(n-1)\pi^{\frac{n}{2}-1}r^{n-2}} + \frac{Q^2}{r^{2n-4}} + \frac{r^2}{l^2}. \tag{4}$$

Here $k = 1, 0, -1$, respectively, correspond to the sphere, plane, and hyperbola symmetric cases. If denoting r_+ as the position of a black hole horizon satisfying $f(r_+) = 0$, one can straightforwardly obtain the value of r_+, with the mass of the black hole within the event horizon radius,

$$M = \frac{(n-1)\pi^{\frac{n}{2}-1}{r_+}^{-n-2}(l^2Q^2{r_+}^4 + l^2{r_+}^{2n}k + {r_+}^{2n+2})}{8l^2\Gamma(\frac{n}{2})}. \tag{5}$$

Correspondingly, the Hawking temperature, entropy, and potential of the black hole could be obtained:

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi}\left(\frac{k(n-2)}{r_+} - (n-2)Q^2{r_+}^{3-2n} - \frac{2r_+\Lambda}{n-1}\right), \tag{6}$$

$$S = \int_0^{r_+} \frac{\partial_{r_+}M(r_+, Q)}{T} = \frac{\pi^{\frac{n}{2}}{r_+}^{n-1}}{2\Gamma\left(\frac{n}{2}\right)}, \tag{7}$$

$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_S = \frac{(n-1)\pi^{\frac{n}{2}-1}Q{r_+}^{2-n}}{4\Gamma\left(\frac{n}{2}\right)}. \tag{8}$$

When taking $k = 1$, from Eq. (6), we will get

$$Q^{\frac{1}{n-2}} = -\frac{n-1}{4\Lambda}A\left(4\pi T - \sqrt{16\pi^2T^2 - \frac{8\Lambda(n-2)}{n-1}\left(A^{2n-4}-1\right)}\right), \tag{9}$$

where $A = \left(\frac{4\Gamma(n/2)}{(n-1)\pi^{n/2-1}}\Phi\right)^{1/(n-2)}$. From the derivative of Eq. (9)

$$\left(\frac{\partial Q}{\partial \Phi}\right)_T = 0, \tag{10}$$

$$\left(\frac{\partial^2 Q}{\partial \Phi^2}\right)_T = 0, \tag{11}$$

we will obtain the following expression:

$$(n-2)(2n-3)A^{2n-2} - (n-2)A^2 - \frac{2\Lambda}{n-1}Q^{2/(n-2)} = 0, \tag{12}$$

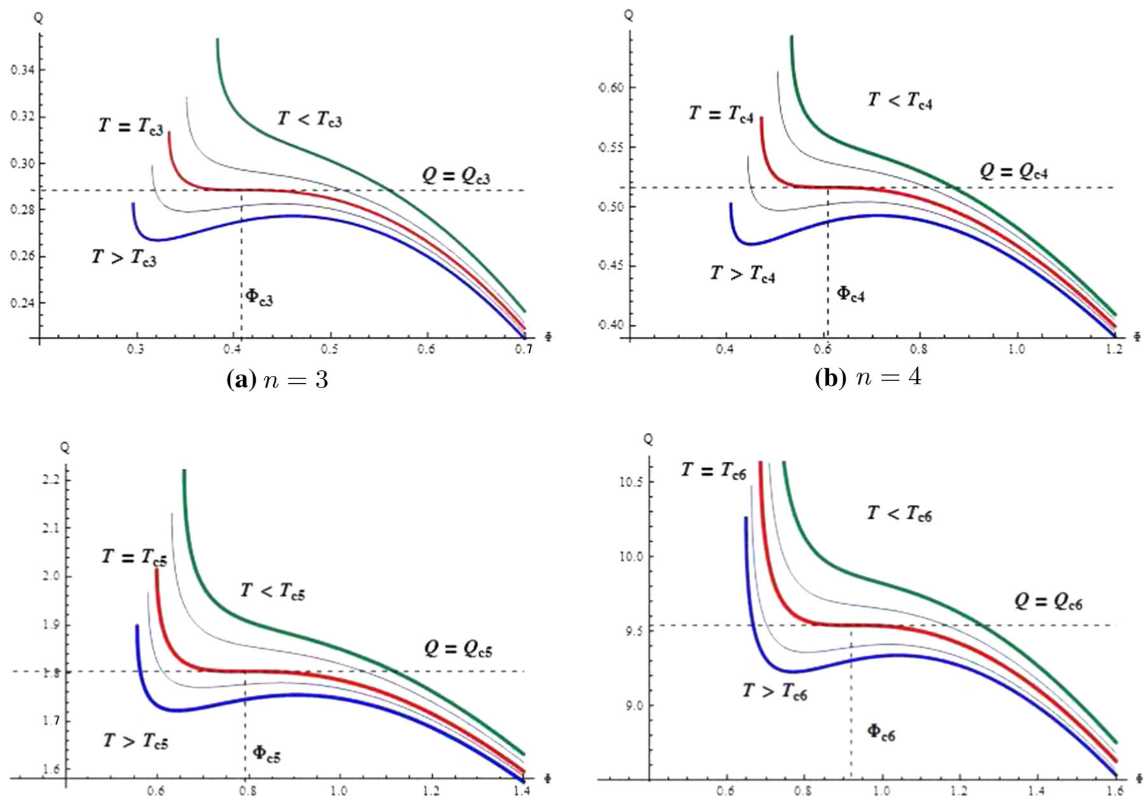
$$(n-1)(2n-3)A^{2n-4} - 1 = 0. \tag{13}$$

Combining Eqs. (9), (12), and (13), one can obtain the three quantities under the critical condition,

$$\Phi_c = \frac{\pi^{n/2-1}}{4\Gamma(n/2)}\sqrt{\frac{(n-1)}{(2n-3)}}, \tag{14}$$

$$T_c = \frac{(n-2)}{\pi(2n-3)}(-2\Lambda)^{1/2}, \tag{15}$$

$$Q_c = \left(-\frac{(n-2)^2}{2\Lambda}\right)^{(n-2)/2}((n-1)(2n-3))^{-1/2}. \tag{16}$$



(a) $n = 3$ **(b)** $n = 4$ **(c)** $n = 5$ **(d)** $n = 6$

Fig. 1 Q–Φ diagram under isothermal conditions when the parameter n is fixed at $n = 3, 4, 5, 6$, respectively

With the critical values for Q, Φ, T given in Eqs. (14)–(16), we derive the critical ratio as

$$\rho_c = \frac{\sqrt{-\Lambda}2^{-\frac{n}{2}-\frac{3}{2}}(n-1)\pi^{n/2}\left(-\frac{(n-2)^2}{\Lambda}\right)^{n/2}}{(n-2)^3\sqrt{\frac{n-1}{2n-3}}\sqrt{2n^2-5n+3}\,\Gamma\left(\frac{n}{2}\right)}. \tag{17}$$

It is obvious that this critical ratio depends on both Λ and n, which is different from the P–V criticality.

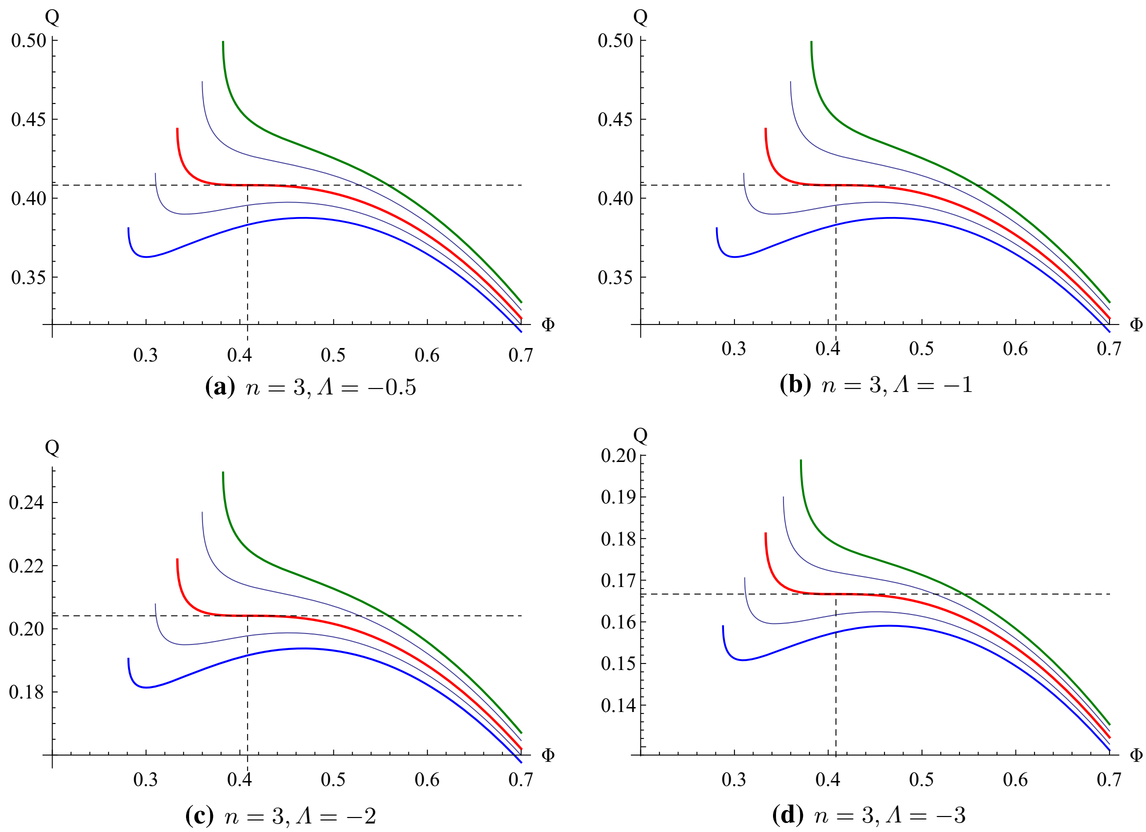
When taking $n = 3, 4, 5, 6$ and $\Lambda = -1$, one can get the Q–Φ graphs at different temperature from the above equations, as well as different critical temperatures under different space-time dimension, respectively $T_{c3} = 0.150053$, $T_{c4} = 0.180063$, $T_{c5} = 0.192925$, and $T_{c6} = 0.20007$. Figure 1 shows the Q–Φ diagram at different critical temperature T_c, from which one can see that Q–Φ curve intersect with x-axis at $\Phi = \Phi_c$. It is apparent that when the temperature $T > T_c$, the special region with $(\frac{\partial Q}{\partial \Phi})_T < 0$ indeed exist in the Q–Φ diagram, which does not satisfy the requirements of thermodynamic stability in the process of black hole evolution. From Fig. 1, we can see that both critical parameters, Q_c and Φ_c, also increase with the dimensionality n, which is similar to the behavior of the critical temperature T_c. In order to perceive the effect of the cosmological constant, we also plot the Q–Φ curves with different values of Λ in Fig. 2. It is shown that, although the position of the critical point shifts with the cosmological constant Λ, this varied parameter does not significantly change the Q–Φ criticality.

Another important thermodynamical quantity is the heat capacity C_Q at constant charge, which measures the stability against a small perturbation and could be defined as

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q = \frac{-\left(2(-1+n)^2\pi^{1+\frac{n}{2}}Q^{\frac{1}{-2+n}}{r_+}^{-1+n}\left(Q{r_+}^{2-n}\right)^{\frac{3}{-2+n}}\right)}{\Delta}T, \tag{18}$$

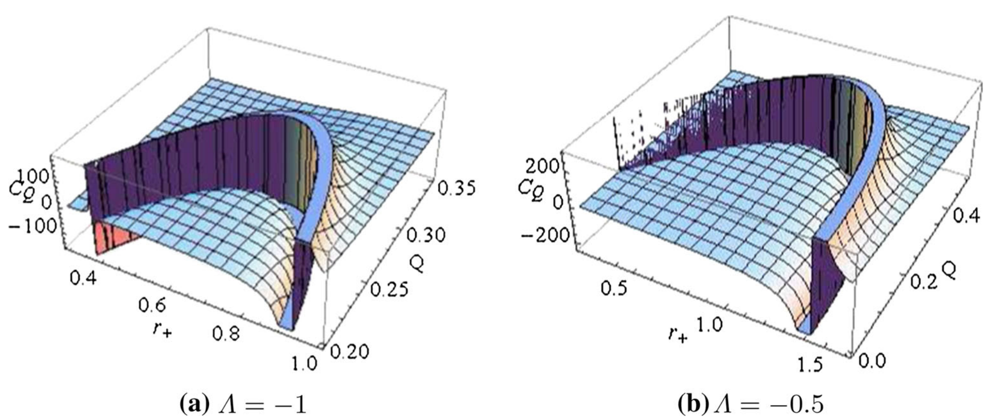
where

$$\Delta = \Gamma\left(\frac{n}{2}\right)(-2n^3\left(\left(Q{r_+}^{2-n}\right)^{\frac{1}{-2+n}}\right)^{2n} + n^2\left(\left(Q{r_+}^{2-n}\right)^{\frac{4}{-2+n}} + 9\left(\left(Q{r_+}^{2-n}\right)^{\frac{1}{-2+n}}\right)^{2n}\right) - n\Gamma\left(\frac{n}{2}\right)\left(3\left(Q{r_+}^{2-n}\right)^{\frac{4}{-2+n}} + 13\left(\left(Q{r_+}^{2-n}\right)^{\frac{1}{-2+n}}\right)^{2n}\right)$$



(a) $n=3, \Lambda=-0.5$ **(b)** $n=3, \Lambda=-1$ **(c)** $n=3, \Lambda=-2$ **(d)** $n=3, \Lambda=-3$

Fig. 2 Q–Φ diagram under isothermal conditions when the parameter $n=3$ is fixed and $\Lambda=-0.5, -1, -2, -3$, respectively



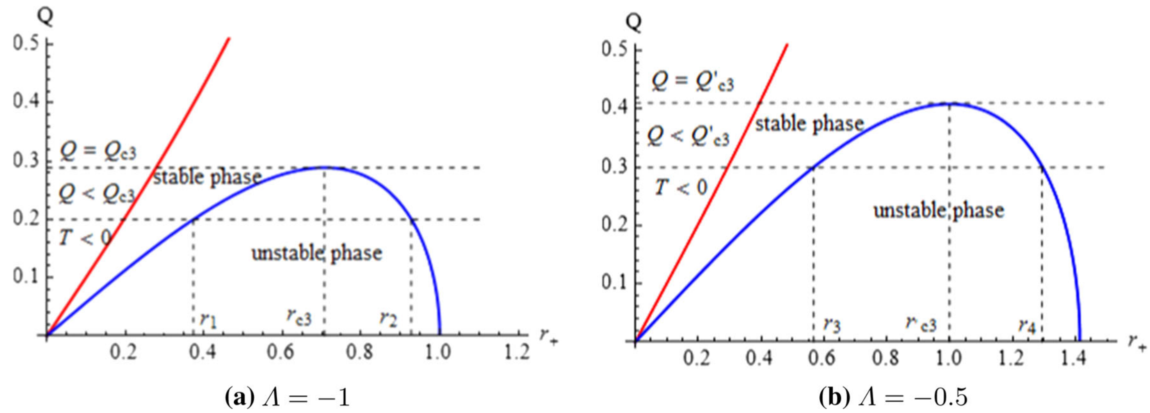
(a) $\Lambda=-1$ **(b)** $\Lambda=-0.5$

Fig. 3 Heat capacity C_Q varying with r_+ and Q. **a** for $n=3$, $\Lambda=-1$, **b** for $n=3$, $\Lambda=-0.5$

$$+2\Gamma\left(\frac{n}{2}\right)\left(\left(Q{r_+}^{2-n}\right)^{\frac{4}{-2+n}}+3\left(\left(Q{r_+}^{2-n}\right)^{\frac{1}{-2+n}}\right)^{2n}\right)$$
$$+2\Gamma\left(\frac{n}{2}\right)Q^{\frac{2}{-2+n}}\left(Q{r_+}^{2-n}\right)^{\frac{2}{-2+n}}\Lambda. \qquad (19)$$

We remark here that C_Q defined in Eq. (18) only reflects the local thermodynamic stability of the black hole. In order to investigate the global thermodynamic stability, we will include the Gibbs free energy in Sect. 4.

For comparison, in Fig. 3 we show the heat capacity C_Q changing with r_+ and Q, fixing $\Lambda=-1$ and $\Lambda=-0.5$ at $n=3$. As can be seen from Fig. 3a, for a small value of Q, with the increase of r_+, the heat capacity first goes to positive infinity at $r_+=r_1$, then increases from negative infinity to a finite negative value and goes back to negative infinity at $r_+=r_2$. Finally, it will decrease from positive infinity to a finite positive value, and then monotonically increases to infinity at $r_+=\infty$. From the above analysis, it is quite



(a) $\Lambda = -1$ **(b)** $\Lambda = -0.5$

Fig. 4 Divergent behavior of the heat capacity as a function of r_+. (a) for $n = 3$, $\Lambda = -1$, (b) for $n = 3$, $\Lambda = -0.5$

evident that the heat capacity C_Q is positive for $r_+ < r_1$ and $r_+ > r_2$, while it is negative for $r_2 < r_+ < r_1$. Note that r_1 represents the transition point where a small stable black hole with $C_Q > 0$ changes to an intermediate unstable one with $C_Q < 0$. r_2 corresponds to the transition point at which an intermediate unstable black hole changes to a large stable one.

On the other hand, for a large value of Q, we find the divergent behavior vanishes and the black hole will stay in a stable phase. In order to obtain a better understanding of the divergent behaviors of the heat capacity C_Q, we illustrate in Fig. 4a the divergent point in the (Q, r_+) plane. We emphasize that $T < 0$ represents a non-black hole case, which will not be discussed here. For $Q < Q_{c3}$, there exist two divergent points at r_1 and r_2; for $Q = Q_{c3}$, the two divergent points coincide with each other at $r_+ = r_{c3}$; while the divergent point disappears at $Q > Q_{c3}$. Therefore, Q_{c3} represents a critical phase transition point, which corresponds to a local maximum along this divergent curve. Moreover, this critical value also varies with the parameter Λ, as can be seen from both Figs. 3b and 4b.

We notice here that, as is shown in Fig. 4, a phase transition occurs when the temperature is higher than the critical temperature, which is different from the case in the P–V curves. Since a higher temperature corresponds to a lower electric charge, this leads to the appearance of a phase transition for small-charge black holes.

3 Phase equilibrium and Maxwell's equal area law

For general $(n + 1)$-dimensional RN-AdS black holes with constant temperature, the Q–Φ curve shows an unstable region with $(\frac{\partial Q}{\partial \Phi})_T < 0$. This is similar to AdS space-time black holes with (T, P, V) as state parameters. However, these problems were all solved when taking the gas to liquid phase transition [19,51,54]. In this section, we will study the famous Maxwell equal area law in the Van der Waals equation, and we apply it to the phase transition of general $(n+1)$-dimensional RN-AdS black holes. Here we choose (T, Q, Φ) to be the state parameters and obtain the two-phase coexistence region boundary. In this region, the isotherm of general $(n + 1)$-dimensional RN-AdS black holes is replaced by an isopotential line. The maximum of the coexistence curve is the so-called critical point, at which Maxwell's equal area law is no longer applicable.

Assuming the temperature is higher than the critical temperature ($T_0 > T_c$), we set the x-axis of the boundary of the two-phase region to be $\Phi_2^{1/(n-2)} = \tilde{\Phi}_2$ and $\Phi_1^{1/(n-2)} = \tilde{\Phi}_1$, while the y-axis is set to be $Q_0^{1/(n-2)} = \tilde{Q}_0$. From Maxwell's equal area law, we have

$$\tilde{Q}_0(\tilde{\Phi}_2 - \tilde{\Phi}_1) = \int_{\tilde{\Phi}_2}^{\tilde{\Phi}_1} \tilde{Q} \mathrm{d}\tilde{\Phi}. \tag{20}$$

Taking $A = \tilde{A}\tilde{\Phi}$ and $\tilde{A} = (\frac{4\Gamma(n/2)}{(n-1)\pi^{n/2-1}})^{1/(n-2)}$, Eq. (9) will transform into

$$\tilde{Q} = -\frac{n-1}{4\Lambda}\tilde{A}\tilde{\Phi} \times \left(4\pi T - \sqrt{16\pi^2 T^2 - \frac{8\Lambda(n-2)}{n-1}\left(\tilde{A}^{2n-4}\tilde{\Phi}^{2n-4} - 1\right)}\right), \tag{21}$$

and the above equation can be rewritten as

$$\tilde{Q} = -\frac{n-1}{4\Lambda} 4\pi T \tilde{A}\tilde{\Phi} + \frac{n-1}{4\Lambda}\tilde{A}\tilde{\Phi}\sqrt{B - C\tilde{\Phi}^{2n-4}}, \tag{22}$$

where $B = 16\pi^2 T^2 + \frac{8\Lambda(n-2)}{(n-1)}$, $C = \frac{8\Lambda(n-2)}{(n-1)}\tilde{A}^{2n-4}$.

Now again, considering a special case with $n = 3$, Eq. (20) turns into

$$Q_0(\Phi_2-\Phi_1)=-\frac{\pi T}{\Lambda}\Phi_2^2-\frac{1}{3\Lambda^2}\left(4\pi^2T^2+\Lambda-\Lambda\Phi_2^2\right)^{\frac{3}{2}}$$
$$+\frac{\pi T}{\Lambda}\Phi_1^2+\frac{1}{3\Lambda^2}\left(4\pi^2T^2+\Lambda-\Lambda\Phi_1^2\right)^{\frac{3}{2}} \tag{23}$$

and

$$Q_0=\left(-\frac{2\pi T}{\Lambda}+\frac{1}{\Lambda}\sqrt{4\pi^2T^2+\Lambda-\Lambda\Phi_2^2}\right)\Phi_2,$$
$$Q_0=\left(-\frac{2\pi T}{\Lambda}+\frac{1}{\Lambda}\sqrt{4\pi^2T^2+\Lambda-\Lambda\Phi_1^2}\right)\Phi_1. \tag{24}$$

From Eq. (24), one can easily obtain the following expressions:

$$0=-2\pi T(\Phi_2-\Phi_1)$$
$$+\Phi_2\sqrt{4\pi^2T^2+\Lambda-\Lambda\Phi_2^2}-\Phi_1\sqrt{4\pi^2T^2+\Lambda-\Lambda\Phi_1^2}, \tag{25}$$

$$2Q_0=-\frac{2\pi T}{\Lambda}(\Phi_2+\Phi_1)$$
$$+\frac{\Phi_2}{\Lambda}\sqrt{4\pi^2T^2+\Lambda-\Lambda\Phi_2^2}+\frac{\Phi_1}{\Lambda}\sqrt{4\pi^2T^2+\Lambda-\Lambda\Phi_1^2}. \tag{26}$$

Then substituting Eq. (26) into Eq. (23), we get

$$(\Phi_2-\Phi_1)\left(-2\pi T(\Phi_2+\Phi_1)\right)$$
$$+(\Phi_2-\Phi_1)(\Phi_2\sqrt{4\pi^2T^2+\Lambda-\Lambda\Phi_2^2})$$
$$+(\Phi_2-\Phi_1)(\Phi_1\sqrt{4\pi^2T^2+\Lambda-\Lambda\Phi_1^2})$$
$$=-2\pi T\Phi_2^2-\frac{2}{3\Lambda}\left(4\pi^2T^2+\Lambda-\Lambda\Phi_2^2\right)^{\frac{3}{2}}$$
$$+2\pi T\Phi_1^2+\frac{2}{3\Lambda}\left(4\pi^2T^2+\Lambda-\Lambda\Phi_1^2\right)^{\frac{3}{2}}. \tag{27}$$

On letting $x=\Phi_1/\Phi_2$, $T=\chi T_c$, and $T_c=\frac{(-2\Lambda)^{1/2}}{3\pi}$ from Eq. (15), the combination of Eqs. (25) and (27) provides us with

$$0=-\frac{2^{\frac{3}{2}}}{3}\chi(1-x)+\sqrt{\frac{8}{9}\chi^2-1+\Phi_2^2}-x\sqrt{\frac{8}{9}\chi^2-1+x^2\Phi_2^2}, \tag{28}$$

$$(1-x)\left(\sqrt{\frac{8}{9}\chi^2-1+\Phi_2{}^2}+x\sqrt{\frac{8}{9}\chi^2-1+x^2\Phi_2{}^2}\right)$$
$$+(1-x)\left(-\frac{2^{\frac{3}{2}}}{3}\chi(1+x)\right)=-\frac{2^{\frac{3}{2}}}{3}\chi(1-x^2)$$
$$-\frac{2}{3\Phi_2^2}\left(\frac{8}{9}\chi^2-1+\Phi_2^2\right)^{\frac{3}{2}}-\frac{2}{3\Phi_2^2}\left(\frac{8}{9}\chi^2-1+x^2\Phi_2^2\right)^{\frac{3}{2}}. \tag{29}$$

From the above formulas, one can see that the value of x and Φ_2 is independent of Λ. For a fixed χ, i.e., a fixed T_0, we can get a certain value for x and Φ_2 from Eqs. (28) and (29).

Substituting Eq. (24) into Eq. (21), one can obtain a similar formula for a high-dimensional space-time,

$$\tilde{Q}=-\frac{n-1}{\sqrt{-2\Lambda}}\tilde{A}\tilde{\Phi}_2$$
$$\times\left(\frac{2(n-2)\chi}{(2n-3)}-\sqrt{\frac{4(n-2)^2\chi^2}{(2n-3)^2}+\frac{n-2}{n-1}\left((\tilde{A}\tilde{\Phi}_2)^{2n-4}-1\right)}\right). \tag{30}$$

Note we can get the $\tilde{Q}_0$ in coexistence region from the above equation. Figure 5 shows the Q–Φ line on the background of isotherms at different temperature. When the temperature is lower than T_c, the isopotential line will replace the curve which does not meet the requirements of thermodynamic stability. The numerical values of χ, x, Φ_1, Φ_2, T_0, and Q_0 at different space-time dimensions are also explicitly illustrated in Table 1.

In the canonical ensemble with fixed charge, the potential, which is also the free energy of the system, presents the thermodynamic behavior of a system in a standard approach. However, in our analysis we will consider an extended phase space. According to the first law of black hole thermodynamics and the interpretation of M (total mass of black hole) [21,56] as H (the black hole enthalpy) [56,57], the Gibbs free energy of the black hole can be written as

$$G=M-TS=\frac{1}{8\Gamma\left(\frac{n}{2}\right)}\pi^{\frac{n}{2}-1}r_+{}^{n-2}\left\{-Q^{\frac{1}{2-n}}r_+\left(Qr_+{}^{2-n}\right)^{\frac{1}{2-n}}\right.$$
$$\times\left((n-2)\left(\left(Qr_+{}^{2-n}\right)^{\frac{2}{n-2}}-\left(Qr_+{}^{2-n}\right)^{\frac{2n-2}{n-2}}\right)\right.$$
$$\left.\left.-\frac{2Q^{\frac{2}{n-2}}\Lambda}{n-1}\right)+(n-1)\left(1+Q^2r_+{}^{4-2n}+\frac{2r_+{}^2\Lambda}{n-n^2}\right)\right\}. \tag{31}$$

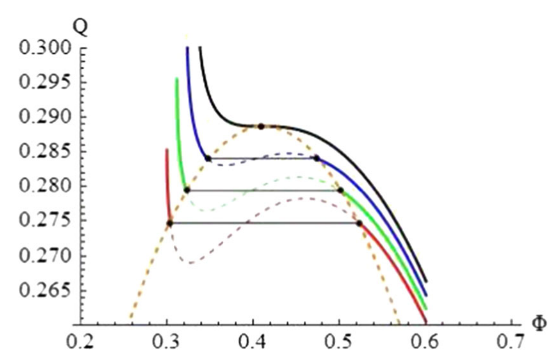
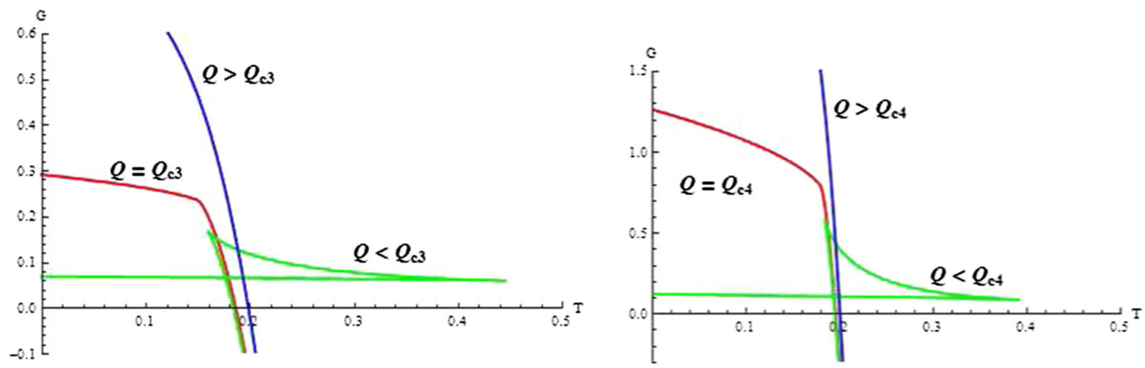


Fig. 5 The simulated isothermal phase transition by isobars and the boundary of two-phase coexistence region for an RN-AdS *black hole*. The boundary of the two-phase equilibrium region is denoted by dotted dashed curve with $n=3$, and $\chi=1$ (*black*), $\chi=1.004$ (*blue*), $\chi=1.008$ (*green*), $\chi=1.012$ (*red*). The area enclosed by *yellow dotted lines* represents the two-phase coexistence region

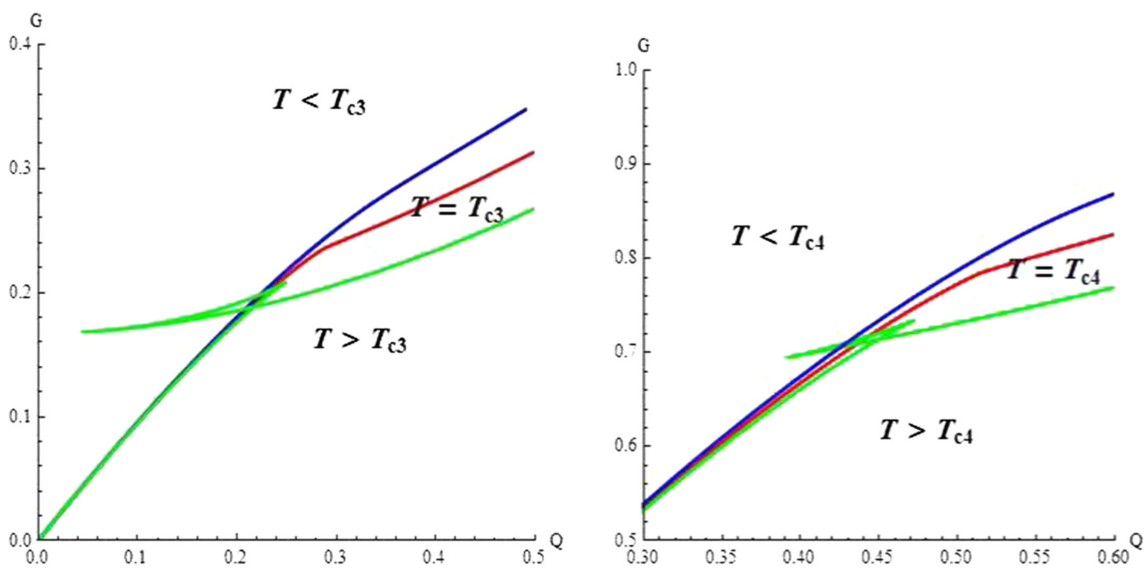
Table 1 State quantities at phase transition points, considering different space-time dimensions ($\Lambda = -1$)

n	χ	x	Φ_1	Φ_2	Q_0	T_0
	1	1	0.408248	0.408248	0.288675	0.150053
	1.002	0.802808	0.364321	0.453809	0.286363	0.150353
3	1.004	0.732417	0.346572	0.473190	0.284047	0.150653
	1.006	0.682245	0.333122	0.488273	0.281726	0.150953
	1	1	0.608376	0.608376	0.516398	0.180063
	1.002	0.643326	0.482451	0.749933	0.503992	0.180423
4	1.004	0.533619	0.434306	0.813887	0.491562	0.180784
	1.006	0.460886	0.398823	0.865339	0.479109	0.181144
	1	1	0.919546	0.919546	0.540560	0.200070
6	1.001	0.534718	0.660438	1.235115	9.159250	0.200270
	1.002	0.408734	0.566763	1.386630	8.778500	0.200470



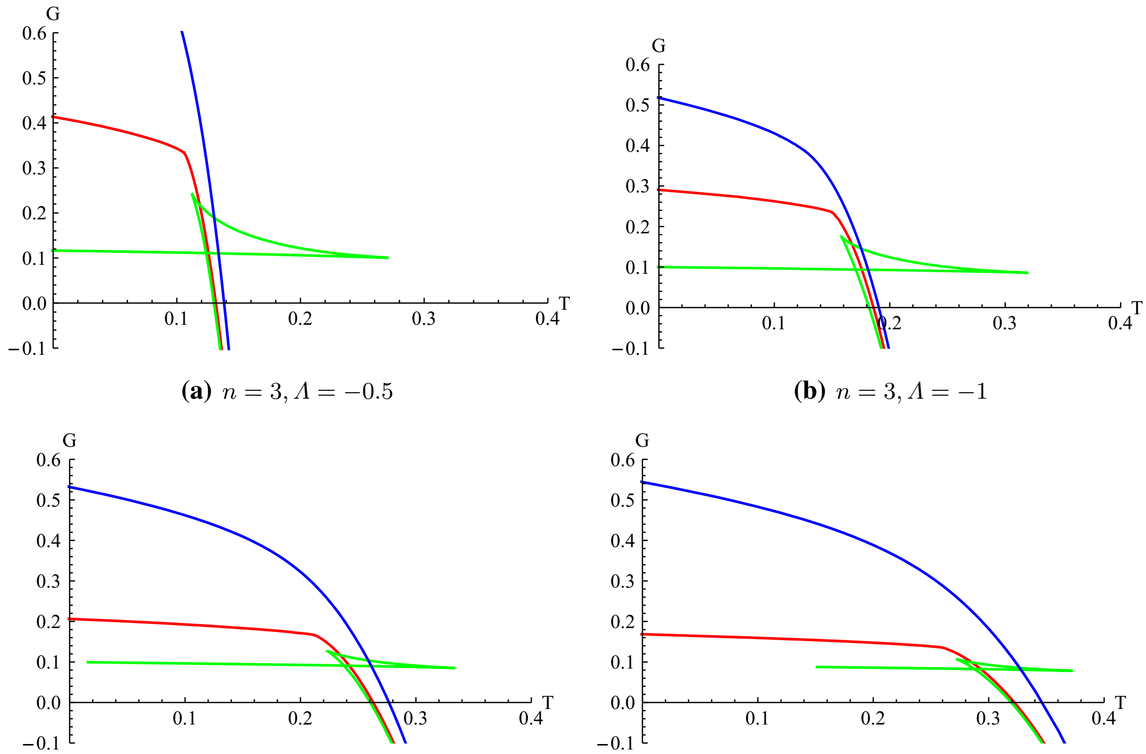
(a) $n = 3$ **(b)** $n = 4$

Fig. 6 Gibbs free energy versus T for **a** $n = 3$, $\Lambda = -1$ and **b** $n = 4$, $\Lambda = -1$



(a) $n = 3$ **(b)** $n = 4$

Fig. 7 Gibbs free energy versus Q for **a** $n = 3$, $\Lambda = -1$ and **b** $n = 4$, $\Lambda = -1$



(a) $n=3, \Lambda=-0.5$ **(b)** $n=3, \Lambda=-1$

(c) $n=3, \Lambda=-2$ **(d)** $n=3, \Lambda=-3$

Fig. 8 Gibbs free energy versus T for $n=3$, and $\Lambda=-0.5,-1,-2,-3$, respectively

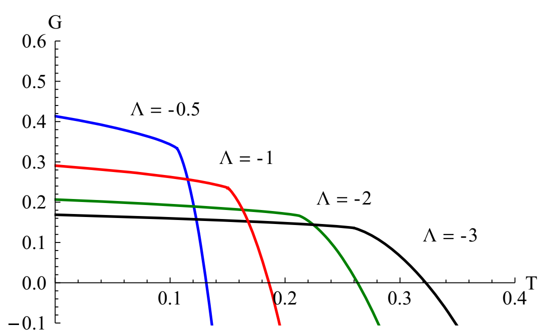


Fig. 9 Gibbs free energy versus $T = Tc$ for $n=3$, and $\Lambda = -0.5, -1, -2, -3$, respectively

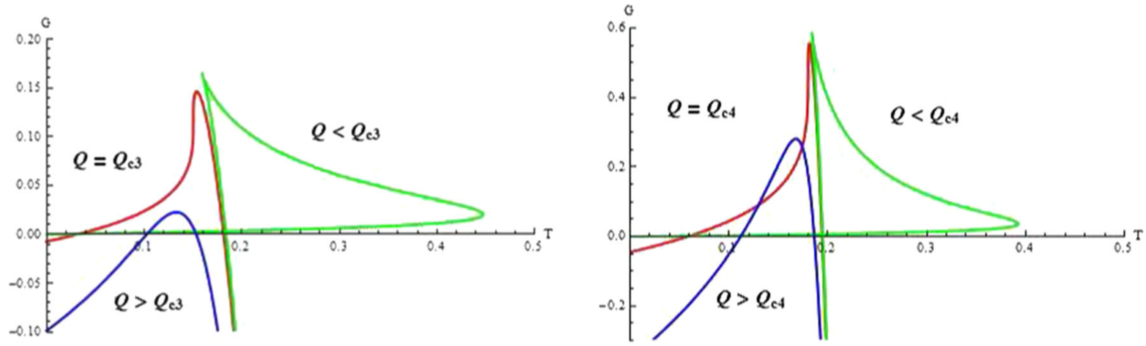
Here, according to Eq. (9), r_+ is a function of charge and temperature, $r_+ = r_+(Q,T)$. In Figs. 6 and 7, we plot the change of the free energy G with T (for fixed Q) and Q (for fixed T) in different space-time dimensions. The existence of a "swallow tail", a behavior first introduced and extensively discussed in Refs. [49,50], is also clearly revealed in our analysis, which indicates that the small–large black hole phase transition occurring in the system is of the first order.

From Figs. 6 and 7, we find that general $(n+1)$-dimensional RN-AdS black holes thermodynamic systems have typical characteristics of Van's system gas/liquid phase transition. In Figs. 8 and 9 we also plot the G–T curves at the same dimension and different values of Λ. It is shown that the G–T criticality does not vary with Λ, while the position of the critical point may change with different values of Λ.

If we do not treat the cosmological constant as a thermodynamic variable and consider the non-extended phase space, the black hole mass M now should be the internal energy of the system and the Gibbs free energy is defined by the following form [38]:

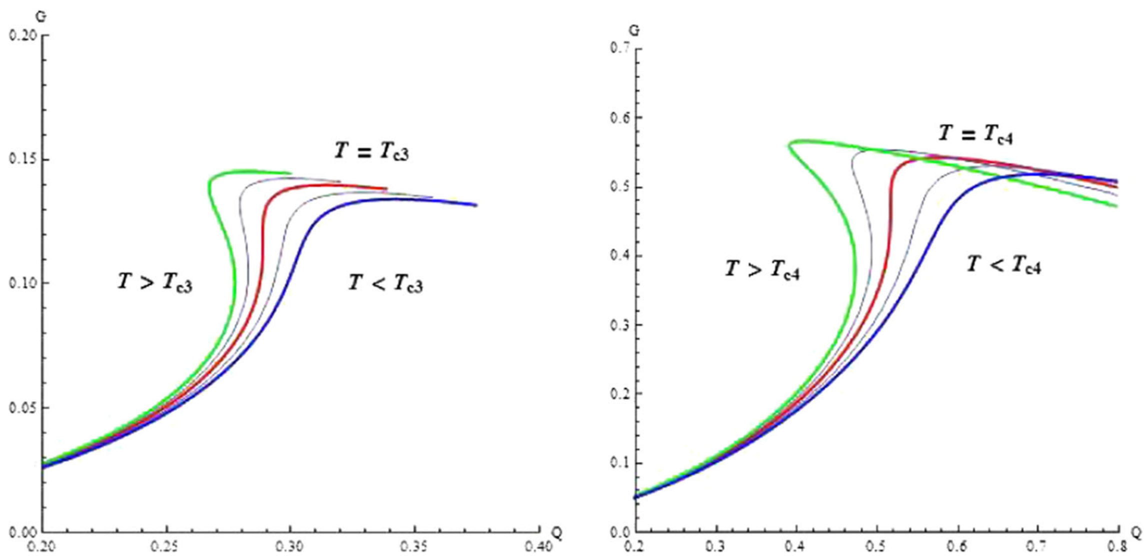
$$
\begin{aligned}
G &= M - TS - Q\Phi \\
&= \frac{(n-1)\,\pi^{\frac{n}{2}-1} {r_+}^{n-2}\left(1 + Q^2 {r_+}^{4-2n} - \frac{2{r_+}^2\Lambda}{(n-1)n}\right)}{8\Gamma(\frac{n}{2})} \\
&\quad - \frac{1}{8\Gamma(\frac{n}{2})}\pi^{\frac{n}{2}-1} Q^{-\frac{1}{n-2}} {r_+}^{n-1}\left(Q{r_+}^{2-n}\right)^{-\frac{1}{n-2}} \\
&\quad \times \Bigg((n-2)\left(Q{r_+}^{2-n}\right)^{\frac{2}{n-2}} - (n-2)\Bigg(\left(Q{r_+}^{2-n}\right)^{\frac{2n-2}{n-2}} \\
&\quad - \frac{2Q^{\frac{2}{n-2}}\Lambda}{n-1}\Bigg) - \frac{(n-1)\,\pi^{\frac{n}{2}-1} Q^2 {r_+}^{2-n}}{4\Gamma(\frac{n}{2})}. \qquad (32)
\end{aligned}
$$

By analyzing the Gibbs free energy of the non-extended phase space, one can detect the differences between the two kinds of phase spaces and furthermore obtain more thermodynamic information of the black hole system. In Figs. 10



(a) $n = 3$ **(b)** $n = 4$

Fig. 10 Gibbs free energy versus T for **a** $n = 3$, $\Lambda = -1$ and for **b** $n = 4$, $\Lambda = -1$



(a) $n = 3$ **(b)** $n = 4$

Fig. 11 Gibbs free energy versus Q for **a** $n = 3$, $\Lambda = -1$ and for **b** $n = 4$, $\Lambda = -1$

and 11 we plot the change of the free energy G with T and Q, for fixed $\Lambda = -1$ and different space-time dimensions. Figure 10 reveals the existence of "swallow tail" behavior of the free energy. However, due to a distinct definition for Gibbs free energy, we fail to detect the "swallow tail" in Fig. 11, which is different from the case shown in Fig. 7.

4 Discussion

Taking general $(n + 1)$-dimensional RN-AdS black holes as thermodynamic systems, the state equation is meaningless in some region. Using Maxwell's equal area law (deduced from the minimum Gibbs free energy theory) and taking a phase transition into consideration, the meaningless region in the state equation no longer exists. Figures 1 and 5 show that, when the system is at constant temperatures higher than the critical temperature, the Q–Φ curves are partially replaced by the isotherm and isopotential lines, which implies Q and T are invariants, while the potential Φ is changing. This region is a two-state coexistence region, where the phase transition is of first order according to the Ehrenfest classification.

From the discussion above, we know that, for a general $(n+1)$-dimensional RN-AdS black hole thermodynamic system, when taking (T, Q, Φ) as state parameters, the system shows similar phase transition characteristics to that of Van's system. The position of the critical point is also the same as the case when taking (T, P, V) as state parameters.

Moreover, by applying Maxwell's equal area law to phase transition behaviors of thermodynamic system, we have derived both the position of the critical point and of the two-phase coexistence region, which makes it possible to obtain a clearer understanding of the phase transition process of such systems [39].

Taking AdS black hole as a thermodynamic system, it was found that the phase transition of various AdS black holes are similar to that of the Van der Waals–Maxwell gas liquid [20,22,27]. Therefore, we can find some observable systems (Van der waals gas) similar to the AdS and dS background black holes. Considering the similarities they share in the thermodynamic properties, we may work backward and investigate other properties of black holes, such as phase transition and critical behaviors. This study will further contribute to a deeper understanding of black hole entropy, temperature, and thermal capacity, as well as the completion of self-consistent black hole thermodynamics.

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